7η διάλεξη

29/10/2018

ΔΙΑΦΟΡΙΚΕΣ ΕΞΙΣΩΣΕΙΣ

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightsquigarrow f(x,y) = C$$

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Οι πιο απλές συναρτήσεις για να έχουμε ολοκληρωτικούς παράγοντες είναι οι εξής:

(i) $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = P(x) \rightarrow \rho(x) = e^{\int P(x)dx}$$

$$\rightarrow \underbrace{e^{\int P(x)dx} M(x,y)dx}_{M_1(x,y)dx} + \underbrace{e^{\int P(x)dx} N(x,y)}_{N_1(x,y)dy} = 0$$

$$\frac{\partial M_1(x,y)}{\partial y} - \frac{\partial N_1(x,y)}{\partial x} =$$

$$= \frac{\partial M}{\partial y} e^{\int P(x)dx} - e^{\int P(x)dx} \cdot P(x) N(x,y) - e^{\int P(x)dx} \frac{\partial N}{\partial x}$$

$$= e^{\int P(x)dx} \left[\frac{\partial M}{\partial y} - \frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} - \frac{\partial N}{\partial x} \right] = 0$$

Παράδειγμα 2, σελ 47)

$$(2xy+y^3)dx + (3x^2+xy^2)dy=0 \quad // \quad p(x,y)=x^m y^n, \quad m,n\in\mathbb{Z}$$

Πολ/ντας με $x^m y^n$

$$\underbrace{(2x^{m+1}y^{n+1}+x^m y^{n+3})}_{M_1}dx + \underbrace{(3x^{m+2}y^n + x^{m+1}y^{n+2})}_{N_1}dy = 0$$

$$\frac{\partial M_1}{\partial y} = 2x^{m+1}(n+1)y^n + x^m(n+3)y^{n+2}$$

$(*)\ x^{m+1}y^n = x^{m+1}y^n$

$$\frac{\partial N_1}{\partial x} = 3(m+2)x^{m+1}y^n + (m+1)x^m y^{n+2}$$

$$\underbrace{\begin{gathered}2(n+1)=3(m+2)\\ n+3=m+1\end{gathered}}$$

$m=-8$

$n=-10$

Για τις τιμές m,n που βρήκαμε έχουμε: $\begin{pmatrix} m=-8 \\ n=-10 \end{pmatrix}$

$$(2x^{-7}y^{-9}+x^{-8}y^{-7})dx + (3x^{-6}y^{-10}+x^{-7}y^{-8})dy=0$$

$$f(x,y)=\int(2x^{-7}y^{-9}+x^{-8}y^{-7})dx + h(y)$$

$$= 2\frac{x^{-6}}{-6}y^{-9} + \frac{x^{-7}}{-7}y^{-7} + h(y)$$

Έτσι προκύπτει ότι: $\frac{\partial f}{\partial y} = 2\frac{x^{-6}}{-6}(-9)x^{-10} + \frac{x^{-7}}{-7}(-7)y^{-8} + h'(y)$

$$\underbrace{= 3x^{-6}y^{-10} + x^{-7}y^{-8}}$$

από εδώ παίρνω ότι $h'(y)=0$

$\Rightarrow$ άρα $h(y)=c_1$ (βοηθ. σταθερά)

Οπότε: $$\underbrace{f(x,y) = 2\frac{x^{-6}}{-6}y^{-9} + \frac{x^{-7}}{-7}y^{-7} + \underset{\substack{\parallel \\ c_1}}{h(y)}}_{\substack{\parallel \\ c}}$$

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$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = P(y) \rightsquigarrow e^{(y)} = e^{\int P(y)dy}$$

όμοιως με (i)

Άσκηση 5, σελ 48 (Βιβλίο) (λύσεις)

$$(xy^2 + y)dx + \underline{\underline{3}}x^2y\,dy = 0$$

Διόρθωση στο βιβλίο: Στο σημείο αυτό αντί για 3 είναι 9!!

Άσκηση 9, σελ 48 (Βιβλίο) (λύσεις)

$$\underbrace{x(x^2+y)}_{\Downarrow}dx + xf(x)dy = 0 \quad , \varrho(x) = x$$

Θα πρέπει: $\frac{\partial}{\partial y}(x^3 + xy) = \frac{\partial}{\partial x}(xf(x))$

$$x = f(x) + xf'(x)$$

παραγ. $\Longrightarrow$ $1 = f'(x) - \frac{1}{x}f(x) \quad , x \neq 0$

$$f(x) = e^{-\int \frac{1}{x}dx}\left[c + \int e^{\frac{1}{x}dx}dx\right]$$

Άσκηση 8, σελ 48 (Βιβλίο)

f: $y^2 \sin x\, dx + y f(x)\, dy = 0$ < f: ολ. ολοκλ. / επιλ.

Θα πρέπει: $\dfrac{\partial(y^2 \sin x)}{\partial y} = \dfrac{\partial(y f(x))}{\partial x}$

$2y\sin x = y f'(x)$ | $f(x) = -2\cos x + C$

$\sin x\, y^2 dx + y(-2\cos x)\, dy = 0$

⋮

(ολοκληρώνω και προχωράω)

Άσκηση 3 i)

$\underbrace{(xy)}_{M} dx + \underbrace{(x^2+y^2+y)}_{N} dy = 0$

$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = x \\ \dfrac{\partial N}{\partial x} = 2x \end{array}\right\}$ $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \dfrac{2x - x}{xy} = \dfrac{x}{xy} = \dfrac{1}{y} = P(y)$

$\rightsquigarrow \; \rho(y) = e^{\int \frac{1}{y} dy} = y.$

Οπότε: $\underbrace{(xy^2)}_{2xy} dx + \underbrace{(x^2 y + y^3 + y^2)}_{2xy} dy = 0$

Αλλαγή Μεταβλητής / Αντικατάστασης:

$$xy' - y\log y = x^2 y$$
$$\Downarrow$$
$$x\frac{y'}{y} - \log y = x^2 \quad , \quad y \neq 0$$
$$\Downarrow$$
$$x(\log y)' - \log y = x^2.$$
$$\Downarrow$$
$$\frac{x(\log y)' - \log y}{x^2} = 1$$
$$\Downarrow$$
$$\left(\frac{\log y}{x}\right)' = 1$$
$$\Downarrow$$
$$\boxed{\log y = (x+c)x}$$

$z = \log y$

$z' = \frac{1}{y} y'$

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- $(y+1)\frac{dy}{dx} + x(y^2+2y) = x \; , \; y(0) = 1 \; , \; z = y^2 + 2y$

$$\Downarrow$$
$$z' = (2y+2)y'$$
$$= 2(y+1)y'$$

$$z' + 2xz = 2x \quad , \quad z(0) = 3$$

$$z(x) = e^{-\int_0^x 2s\,ds}\left[3 + \int_0^x 2s\, e^{\int_0^s 2u\,du}\,ds\right] \quad \left(y^2 + 2y - z = 0 \rightarrow y\right)$$

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Άσκηση 12, σελ 55) (λύσεις)

$$z = x^2 + y^2 \rightsquigarrow z' = 2x + 2yy'$$

$$2yy' = e^{(x^2+y^2)/x} + (x^2+y^2)/x - 2x$$

$$z' - 2x = e^{z/x} + \frac{z}{x} - 2x$$

Θέτω $u = \frac{z}{x} \Rightarrow z = ux \Rightarrow z' = u'x + u$

Έτσι $u'x + \cancel{u} = e^u + \cancel{u} \Rightarrow u'x = e^u$

$$\frac{du}{e^u} = \frac{dx}{x}$$

Άσκηση

$$q(x)y' = yq'(x) - y^2 \ , \ y(0)=1 \ , \ q \in C[\mathbb{R},(0,+\infty)) \quad q(0)=1$$

$$\frac{q(x)y' - yq'(x)}{y^2} = -1$$

$$\Rightarrow \frac{yq'(x) - q(x)y'}{y^2} = 1 \quad \Big| \quad \left(\frac{q(x)}{y}\right)' = 1 \ //$$

Άσκηση 22)

$$\underbrace{\frac{1}{y^2+1}y'}_{=(\operatorname{Arctan} y)'} + \frac{2}{x}\operatorname{Arctan} y = \frac{2}{x}$$

$$(\operatorname{Arctan} y)' + \frac{2}{x}\operatorname{Arctan} y = \frac{2}{x} \xRightarrow{\text{πολ/ζω με } x^2}$$

$$x^2(\operatorname{Arctan} y)' + 2x \operatorname{Arctan} y = 2x$$

$$(x^2 \operatorname{Arctan} y)' = 2x$$

(*) $y'' = F(x, y') \ , \ y' = z$

$z' = F(y, z)$

π.χ

$$xy'' = y' + x^2$$

$$xz' = z + x^2$$

$$z' - \frac{1}{x}z = x$$

$$z(x) = e^{\int \frac{1}{x}dx}\left[c + \int x e^{-\int \frac{1}{x}dx} dx\right]$$

$$z(x) = e^{\ln|x|}\left[c + \int x e^{-\ln|x|} dx\right]$$

$$y(x) = \int z(x) + C$$

$$y(x) = \int |x| \left[c + \int x \frac{1}{|x|} dx\right] + C_1$$

Εξισώσεις όπου λείπει η ανεξάρτητη μεταβλητή:

$$y'' = F(y, y')$$

$\rightsquigarrow$ Θέτω $y' = z \quad || \quad y'' = \frac{dy'}{dx} = \frac{dy'}{dy} \cdot \overbrace{\frac{dy}{dx}}^{=y'} = \frac{dz}{dy} \cdot z$

$$z \cdot \frac{dz}{dy} = F(y, z)$$

$$\frac{dz}{dy} = \frac{F(y,z)}{z}$$

$$\left.\begin{array}{l} y'' = 2yy' \\ y' = z \Rightarrow y'' = z \frac{dz}{dy} \end{array}\right\} \cancel{z} \frac{dz}{dy} = 2y\cancel{z}$$

$$\frac{dz}{dy} = 2y$$

$$dz = 2y\,dy$$

$$z = y^2 + c$$

$$\boxed{y' = y^2 + c}$$

Άρα $\frac{y'}{y^2+c} = 1$

$\Rightarrow \ln|y^2 + c| = x + c_1$

$\Rightarrow y^2 + c = \pm e^{x+c_1}$

$\Rightarrow y = \pm \sqrt{e^{x+c_1} - c}$